

On the Stanley depth of a special class of Borel type ideals

Mircea Cimpoeaş

Abstract

We give sharp bounds for the Stanley depth of a special class of ideals of Borel type. In particular, given a graphic sequence D , using the Havel-Hakimi process, we associate a monomial ideal of Borel type, denoted by $I(D)$, and we give bounds for its Stanley depth.

Keywords: ideal of Borel type, Stanley depth, graphic sequence, Havel-Hakimi theorem.

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Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a \mathbb{Z}^n -graded S -module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ of \mathbb{Z}^n -graded K -vector spaces, where $m_i \in M$ is homogeneous with respect to \mathbb{Z}^n -grading, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$ -submodule of M . We define $\text{sdepth}(\mathcal{D}) = \min_{i=1, \dots, r} |Z_i|$ and $\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}(M)$ is called the *Stanley depth* of M .

Stanley conjectured in [21] that $\text{sdepth}(M) \geq \text{depth}(M)$ for any M . The conjecture was disproved in [11] for $M = S/I$, where $I \subset S$ is a monomial ideal, but remains open in the case $M = I$. Herzog, Vladioiu and Zheng showed in [18] that $\text{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [20], Rinaldo gave a computer implementation for this algorithm, in the computer algebra system CoCoA [10]. For a friendly introduction in the thematic of Stanley depth, we refer the reader to [15].

We say that a monomial ideal $I \subset S$ is of *Borel type*, see [17], if it satisfies the following condition: $(I : x_j^\infty) = (I : (x_1, \dots, x_j)^\infty)$, $(\forall) 1 \leq j \leq n$. The *Mumford-Castelnuovo regularity* of I is the number $\text{reg}(I) = \max\{j - i : \beta_{ij}(I) \neq 0\}$, where β_{ij} 's are the graded Betti numbers. The regularity of the ideals of Borel type was extensively studied, see for instance [17], [3], [1] and [7]. In the first section, we study the invariant $\text{sdepth}(I)$, for an ideal of Borel type. In the general case, we note some bounds for $\text{sdepth}(I)$, see Proposition 1.2 and we give some tighter ones, when I has a special form, see Theorem 1.6.

In the second section, we consider $D = \{d_1, \dots, d_n\}$, a non-increasing sequence of non-negative integers. The sequence D is called *graphic*, if there exists a simple graph $G = (V, E)$ on the set of vertices $V = [n] = \{1, \dots, n\}$ and the set of edges E , such that $D = D(G)$, where $D(G)$ is the sequence of degrees of each vertex of G in non-increasing order. Havel [14] and Hakimi [13] proved that a sequence $D = \{d_1, \dots, d_n\}$ is graphic, if and only if the sequence $D^1 := \{d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n\}$ is graphic, after reordering its elements in a non-increasing sequence. D^1 is called the *Havel-Hakimi derivative* of D .

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The *Havel-Hakimi process*, works as follows: Let D be a graphic sequence. Let D^1 be the Havel-Hakimi derivative of D . For $2 \leq m \leq m$, let D^i be the Havel-Hakimi derivative of D^{i-1} , where $m \leq n-1$ is the first index for which D^m is the empty set or the largest element of D^m is non-positive. As a consequence of the Havel-Hakimi Theorem, the sequence D is graphic if and only if D^m contains only zeroes. The number of zeroes in D^m is called the *residue* of D and is denoted by $R(D)$, see [12] for further details.

I. Anwar, A. Khalid and H. Mahmood [2] introduced a class of Borel type ideals, associated to graphs, called the *elimination ideals*, and studied their stability properties and regularity. See also [1] for further details on the stability of ideals of Borel type. In our paper, we introduce a similar construction: given a graphic sequence D , we associate a monomial ideal $I(D) \subset S$, which we call the *Havel-Hakimi ideal*, see Definition 2.1. We note that $I(D)$ is an ideal of Borel type, see Proposition 2.2, and, using the results from the first section, we compute several algebraic invariants of it, see Theorem 2.3. We conclude our paper with a list of examples, see Example 2.5.

1 Stanley depth of ideals of Borel type

First, we recall the construction of the sequential chain associated to a Borel type ideal $I \subset S$, see [17] for more details. Assume that $\text{Ass}(S/I) = \{P_0, \dots, P_m\}$ with $P_i = (x_1, \dots, x_{n_i})$, where $n \geq n_0 > n_1 > \dots > n_m \geq 1$. Also, assume that $I = \bigcap_{i=0}^m Q_i$ is the reduced primary decomposition of I , with $P_i = \sqrt{Q_i}$, for all $0 \leq i \leq m$.

We define $I_k := \bigcap_{j=k}^m Q_j$, for all $0 \leq k \leq m$. One can easily check that $I_i = (I_{i-1} : x_{n_{i-1}}^\infty)$, for all $1 \leq i \leq m$. The sequence of ideals $I = I_0 \subset I_1 \subset \dots \subset I_m \subset I_{m+1} := S$ is called the *sequential sequence* of I . Let J_i be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \dots, x_{n_i}]$, for all $0 \leq i \leq m$. Then, the saturation $J_i^{\text{sat}} = (J_i : (x_1, \dots, x_{n_i})^\infty) = J_{i+1}S_i$, for all $0 \leq i \leq m$, where $J_{m+1} := S_{m+1}$. One has $I_{i+1}/I_i \cong (J_i^{\text{sat}}/J_i)[x_{n_{i+1}}, \dots, x_n]$. If $M = \bigoplus_{t \geq 0} M_t$ is an Artinian graded S -module, we denote $s(M) = \max\{t : M_t \neq 0\}$. We recall the following result.

Proposition 1.1. ([17, Corollary 2.7]) $\text{reg}(I) = \max\{s(J_0^{\text{sat}}/J_0), \dots, s(J_m^{\text{sat}}/J_m)\} + 1$.

Proposition 1.2. *With the above notations, the following assertions hold:*

- (1) $\text{sdepth}(S/I_i) = \text{depth}(S/I_i) = n - n_i$, for all $0 \leq i \leq m$.
- (2) $\text{sdepth}(I_0) \leq \text{sdepth}(I_1) \leq \dots \leq \text{sdepth}(I_m)$.
- (3) $\text{depth}(I_i) = n - n_i + 1 \leq \text{sdepth}(I_i) \leq \text{sdepth}(P_i) = n - \lfloor \frac{n_i}{2} \rfloor$, $(\forall) 0 \leq i \leq m$.

Proof. (1) Since I_i is an ideal of Borel type for all $0 \leq i \leq m$, it is enough to consider the case $i = 0$. Also, by [18, Lemma 3.6], we can assume $n_0 = n$. Since $P_0 = (x_1, \dots, x_n) \in \text{Ass}(S/I)$, it follows that $\text{depth}(S/I) = 0$ and thus, by [8, Theorem 1.4], $\text{sdepth}(S/I) = 0$. See also [15, Proposition 18].

(2) Since $I_i = (I_{i-1} : x_{n_{i-1}}^\infty)$, by [6, Proposition 2.7], we get $\text{sdepth}(I_{i-1}) \leq \text{sdepth}(I_i)$, for all $1 \leq i \leq m$.

(3) Since $I_i = J_i S$, by [18, Lemma 3.6], it follows that $\text{sdepth}(I_i) = n - n_i + \text{sdepth}_{S_i}(J_i) \geq n - n_i + 1$. Since $P_i \in \text{Ass}(I_i)$, it follows that there exists a monomial $v \in S$ such that

$I_i = (P_i : v)$. Therefore, by [6, Proposition 2.7], it follows that $\text{sdepth}(P_i) \geq \text{sdepth}(I_i)$. On the other hand, P_i is generated by variables. Thus, for example by [9, Theorem 1.3], it follows that $\text{sdepth}(P_i) = n - \left\lfloor \frac{ht(P_i)}{2} \right\rfloor = n - \left\lfloor \frac{n_i}{2} \right\rfloor$. \square

The following result is well known, but we present the sketch of the proof.

Lemma 1.3. *Let $r \leq n$ and a_1, \dots, a_r be some positive integers. If $Q = (x_1^{a_1}, \dots, x_r^{a_r}) \subset S$, then $\text{reg}(Q) = a_1 + \dots + a_r - r + 1$.*

Proof. Let $\bar{Q} = Q \cap S' \subset S'$, where $S' = K[x_1, \dots, x_r]$. As a particular case of Proposition 1.1, we get $\text{reg}(Q) = \text{reg}(\bar{Q}) = s(S'/\bar{Q}) + 1 = a_1 + \dots + a_r - r + 1$. \square

We recall the following result from [3].

Proposition 1.4. ([3, Corollary 3.17]) *If $I \subset S$ is an ideal of Borel type with the irredundant irreducible decomposition $I = \bigcap_{i=1}^r C_i$, then $\text{reg}(I) = \max\{\text{reg}(C_i) : 1 \leq i \leq r\}$.*

Let $n \geq n_0 > n_1 > \dots > n_m \geq 1$ be some integers. Let a_{ij} be some positive integers, where $0 \leq i \leq m$ and $1 \leq j \leq n_i$. We consider the monomial irreducible ideals $Q_i = (x_1^{a_{i1}}, \dots, x_{n_i}^{a_{in_i}})$, for $0 \leq i \leq m$. Let $I_i := \bigcap_{j=i}^m Q_j$ and denote $I = I_0$. Since $P_i = (x_1, \dots, x_{n_i}) = \sqrt{Q_i}$ for all $0 \leq i \leq m$, by [16, Proposition 5.2] or [7, Corollary 1.2], it follows that I is an ideal of Borel type. As a direct consequence of Lemma 1.3 and Proposition 1.4, we get the following corollary.

Corollary 1.5. *If $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$ and $i < m$, then $\text{reg}(I_i) = \text{reg}(Q_i) = a_{i1} + a_{i2} + \dots + a_{in_i} - n_i + 1$, for all $0 \leq i \leq m$.*

With the above notation, we have the following result.

Theorem 1.6. *If $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$ and $i < m$, then for all $0 \leq i \leq m$, it holds that*

$$n - \left\lfloor \frac{n_i}{2} \right\rfloor \geq \text{sdepth}(I_i) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i.$$

Proof. The first inequality follows from Proposition 1.2(3). In order to prove the second one, we use induction on $i \leq m$. If $i = m$, then $I_m = Q_m$ is an irreducible ideal, and therefore, by [9, Theorem 1.3], $\text{sdepth}(I_m) = n - \left\lfloor \frac{n_m}{2} \right\rfloor = n + \left\lceil \frac{n_m}{2} \right\rceil - n_m$.

Assume $i < m$. We can write $Q_i = U_i + V_i$, where $U_i = (x_1^{a_{i1}}, \dots, x_{n_{i+1}}^{a_{in_{i+1}}})$ and $V_i = (x_{n_{i+1}+1}^{a_{i,n_{i+1}+1}}, \dots, x_{n_i}^{a_{in_i}})$. Since $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$, it follows that $U_i \subset Q_{i+1}$. Therefore, $I_i = (U_i + V_i) \cap I_{i+1} = (U_i \cap I_{i+1}) + (V_i \cap I_{i+1})$. Note that $J := U_i \cap I_{i+1} = U_i \cap I_{i+2}$ is a Borel type ideal with the irreducible irredundant decomposition $J = U_i \cap Q_{i+2} \cap \dots \cap Q_m$, and, therefore, of the same class as I_{i+1} . Thus, by induction hypothesis, it follows that $\text{sdepth}(J) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_{i+1}$.

On the other hand, by [6, Remark 1.3] and the induction hypothesis, $\text{sdepth}(V_i \cap I_{i+1}) \geq \text{sdepth}(V_i) + \text{sdepth}(I_{i+1}) - n = \text{sdepth}(I_{i+1}) - \left\lfloor \frac{n_i - n_{i+1}}{2} \right\rfloor$.

Let $\bar{V}_i \subset S' = K[x_{n_{i+1}+1}, \dots, x_{n_i}]$ be the monomial ideal generated by $G(V_i)$ and let $\bar{J} \subset S'' = K[x_1, \dots, x_{n_{i+1}}, x_{n_{i+1}+1}, \dots, x_n]$ be the monomial ideal generated by $G(J)$. Since

$J \subset I_{i+1}$, it follows that $I_i = (\bar{J} \otimes_K (S''/\bar{V}_i)) \oplus (V_i \cap I_i)$. By [5, Proposition 2.10] and [19, Lemma 2.2], we get: $\text{sdepth}(I_i) \geq \min\{\text{sdepth}(J) - n_i + n_{i+1}, \text{sdepth}(I_{i+1}) - \lfloor \frac{n_i - n_{i+1}}{2} \rfloor\} \geq n + \lfloor \frac{n_m}{2} \rfloor - n_i$, as required. \square

Question: What can we say about the case when the condition $a_{ij} \geq a_{i+1,j}$ is removed? Of course, the method used in the proof of the previous Theorem do not work. However, our computer experiments in **Cocoa** [10] suggested that the conclusion of the Theorem 1.6 might be true. Unfortunately, we are not able to give either a proof, or a counterexample.

The next example shows that the bounds given in Theorem 1.6 are sharp.

Example 1.7. Let $I = Q_0 \cap Q_1$, where $Q_0 = (x_1^3, x_2^2, x_3^2, x_4, x_5)$ and $Q_1 = (x_1, x_2, x_3, x_4)$ are ideals in $S = K[x_1, \dots, x_5]$. Then $I_1 = Q_1$ and $\text{sdepth}(I_1) = 5 - \lfloor \frac{4}{2} \rfloor = 3$. Also $n = 5$, $n_0 = 5$ and $n_1 = 4$. Using **CoCoA**, we get $\text{sdepth}(I) = 2 = n - \lfloor \frac{n_1}{2} \rfloor - n_0$. Let $Q'_0 = (x_1^2, x_2^2, x_3, x_4, x_5) \subset S$ and $I' = Q'_0 \cap Q_1$. Using **CoCoA** [10], we get $\text{sdepth}(I') = 3 = n - \lfloor \frac{n_0}{2} \rfloor$.

2 The Havel-Hakimi monomial ideal

Given a sequence $A = \{a_1, \dots, a_j\}$ of non-negative integers, where $1 \leq j \leq n$, we denote by $Q(A)$ the irreducible monomial ideal $(x_1^{a_1+1}, \dots, x_j^{a_j+1}) \subset S$. Let $D = \{d_1, \dots, d_n\} := D^0$ be a graphic sequence. We consider the Havel-Hakimi derivatives D^1, \dots, D^m , where $m \leq n-1$ such that D^m contains only zeroes. We denote $m(D) := m$. We give the following definition, which is similar to [2, Definition 3.2].

Definition 2.1. The Havel-Hakimi ideal associated to D is $I(D) := \bigcap_{i=0}^m Q(D^i)$.

Proposition 2.2. $I := I(D)$ is an ideal of Borel type. Moreover, if $I = I_0 \subset I_1 \subset \dots \subset I_m \subset I_{m+1} := S$ is the sequential chain of I , then $I_i = I(D^i)$, for all $0 \leq i \leq m$.

Proof. Note that $\text{Ass}(S/I) = \{P_0, \dots, P_m\}$, where $P_i = (x_1, \dots, x_{n-i}) = \sqrt{Q(D^i)}$ for all $0 \leq i \leq m$. By [16, Proposition 5.2] or [7, Corollary 1.2], it follows that I is of Borel type. Also, $I_j = \bigcap_{i=0}^m Q(D^i) = I(D^j)$. \square

Note that $I(D)$ is an ideal of the same type as the ideals described in the second part of the first section. Using the results from the first section, we prove the following theorem.

Theorem 2.3. For any graphic sequence D , we have:

- (1) $\dim(S/I(D)) = m(D) = n - R(D)$.
- (2) $\text{sdepth}(S/I(D)) = \text{depth}(S/I(D)) = 0$.
- (3) $\text{pd}(S/I(D)) = n$.
- (4) $\text{reg}(I(D)) = d_1 + d_2 + \dots + d_n + 1$.
- (5) $\text{sdepth}(I(D)) \leq \text{sdepth}(I(D^1)) \leq \dots \leq \text{sdepth}(I(D^{m(D)})) = \lfloor \frac{n+m(D)}{2} \rfloor$.
- (6) $\lfloor \frac{n+i}{2} \rfloor \geq \text{sdepth}(I(D^i)) \geq \lfloor \frac{n-m(D)}{2} \rfloor + i$, for all $0 \leq i < m(D)$.

Proof. (1) Denote $m=m(D)$. Since $\text{Min}(S/I(D)) = \{P_m\}$ and $\text{ht}(P_m) = n - m$, it follows that $\dim(S/I(D)) = m$. On the other hand, $R(D) = n - m(D)$, and thus we are done.

(2) It follows from Proposition 1.2(1).

(3) Is a direct consequence of (2) and the Auslander-Buchsbaum's formula.

(4) Is a particular case of Corollary 1.5.

(5) Denote $I_i = I(D^i)$, for all $0 \leq i \leq m$. Note that $I(D) = I_0 \subset I_1 \subset \cdots \subset I_m \subset I_{m+1} := S$ is the sequential sequence of $I(D)$. Therefore, by Proposition 1.2(2), we get $\text{sdepth}(I(D)) \leq \text{sdepth}(I(D^1)) \leq \cdots \leq \text{sdepth}(I(D^m))$.

On the other hand, since $I_m = Q(D^m)$ is a prime monomial ideal, by [9, Theorem 1.3], it follows that $\text{sdepth}(I(D^m)) = n - \left\lfloor \frac{R(D)}{2} \right\rfloor = \left\lceil \frac{n+m(D)}{2} \right\rceil$.

(6) Since $n_i = n - i$, for all $0 \leq i \leq m$, we are done by Theorem 1.6. \square

Remark 2.4. Note that the results of Theorem 1.6 can be easily applied for ideals of the form $I_{\leq j} = \bigcap_{i=0}^j Q(D^i)$, where $0 \leq j \leq m$. In this case, we have $\dim(S/I_{\leq j}) = n - j$, $\text{reg}(I_{\leq j}) = \text{reg}(Q(D))$ and $\text{sdepth}(I_{\leq j}) \geq \left\lceil \frac{n-j}{2} \right\rceil$.

For a non-negative integer a , we denote by a^k the sequence a, \dots, a of length k . For example, $\{3, 1^3\} = \{3, 1, 1, 1\}$. The bounds given in Theorem 2.3(6) are sharp. Note that the ideals considered in the Example 1.7 are $I = I(\{2, 1^2, 0^2\})$ and $I' = I(\{1^2, 0^3\})$.

We conclude our paper with a list of examples, from graph theory.

Example 2.5. (1) Let G be the *discrete graph* on the vertex set $[n]$, i.e. G has no edges. The degree sequence of G is $D = D(G) = \{0^n\}$. It follows that $I(D) = (x_1, x_2, \dots, x_n)$. Note that $\text{sdepth}(I(D)) = \left\lceil \frac{n}{2} \right\rceil$, see [4, Theorem 2.2].

(2) Let G be the *star graph* on the vertex set $[n]$, i.e. $E(G) = \{\{1, 2\}, \{1, 3\}, \dots, \{1, n\}\}$. Then $D = D(G) = \{n, 1^{n-1}\}$. Note that $D^1 = \{0^{n-1}\}$. Thus $I(D) = (x_1^{n+1}, x_2^2, \dots, x_n^2) \cap (x_1, \dots, x_{n-1}) = (x_1^{n+1}, x_2^2, \dots, x_{n-1}^2, x_n^2 x_1, \dots, x_n^2 x_{n-1})$. By Theorem 2.3, $\dim(R/I(D)) = 1$ and $\text{reg}(I(D)) = 3n$. Moreover, if $n = 2k$, then, $\text{sdepth}(I(D)) = k$ and, if $n = 2k + 1$, then $k \leq \text{sdepth}(I(D)) \leq k + 1$. Using *Cocoa*, for $n = 3$ and $n = 5$, we get $\text{sdepth}(I(D)) = 2$. We conjecture that $\text{sdepth}(I(D)) = k$, for all $n = 2k + 1$ with $k \geq 2$.

(3) Let K_n be the *complete graph* on the vertex set $[n]$. The degree sequence of K_n is $D = D(K_n) = \{(n-1)^n\}$. One can easily check that $m(D) = n - 1$ and $D^i = D(K_{n-i})$ for all $0 \leq i \leq n - 1$, where $D^0 = D$. Therefore, $I(D) = \bigcap_{i=0}^{n-1} (x_1^{n-i}, \dots, x_{n-i}^{n-i})$. By Theorem 2.3, we get $\dim(R/I(D)) = n - 1$ and $\text{reg}(I(D)) = n^2 + 1$. Our computer experiments yield us to conjecture that $\text{sdepth}(I(D)) = \left\lceil \frac{n}{2} \right\rceil$, for all $n \geq 1$.

(4) Let P_n be the *path graph* of length n , i.e. $E(P_n) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$. The degree sequence of P_n is $D = D(P_n) = \{2^{n-2}, 1^2\}$. We consider three cases:

(a) If $n = 3k$, then $D^0 = D = \{2^{3k-2}, 1^2\}$, $D^1 = \{2^{3k-5}, 1^4\}, \dots, D^{k-1} = \{2, 1^{2k}\}$, $D^k = \{1^{2k-2}, 0^2\}$, $D^{k+1} = \{1^{2k-4}, 0^3\}, \dots, D^{2k-1} = \{0^{k+1}\}$.

(b) If $n = 3k + 1$, then $D^0 = D = \{2^{3k-1}, 1^2\}$, $D^1 = \{2^{3k-4}, 1^4\}, \dots, D^{k-1} = \{2^2, 1^{2k}\}$, $D^k = \{1^{2k}, 0\}$, $D^{k+1} = \{1^{2k-2}, 0^2\}, \dots, D^{2k} = \{0^{k+1}\}$.

(c) If $n = 3k + 2$, then $D^0 = D = \{2^{3k}, 1^2\}$, $D^1 = \{2^{3(k-1)}, 1^4\}, \dots, D^k = \{1^{2k+2}\}$, $D^{k+1} = \{1^{2k}, 0\}, \dots, D^{2k+1} = \{0^{k+1}\}$.

Therefore, by Theorem 2.3, $\dim(S/I(D)) = n - \lfloor n/3 \rfloor$ and $\operatorname{reg}(I(D)) = 2n - 1$.

(5) Let C_n be the *cycle graph* of length n , i.e. $E(C_n) = E(P_n) \cup \{n, 1\}$. The degree sequence of C_n is $D = D(C_n) = \{2^n\}$. Note that $D^1 = \{2^{n-3}, 1^2\} = D(P_{n-1})$ therefore, by (4), $\dim(S/I(D)) = n - R(D) = n - \lfloor (n-1)/3 \rfloor$. Also, by Theorem 2.3, $\operatorname{reg}(I(D)) = 2n + 1$.

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Mircea Cimpoeaş, Simion Stoilow Institute of Mathematics, Research unit 5, P.O.Box 1-764,
 Bucharest 014700, Romania
 E-mail: mircea.cimpoeas@imar.ro